EXPLORING THE DIRECTIONS OF THEORY BUILDING IN MATHEMATICS EDUCATION

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Abstract

Considering some implicit assumptions concerning people, such as an unspecified majority, are mathematics education theories consistently effective only in some specific fields, and is their growth also limited? Using case studies in which the authors were involved—criteria for students' knowing geometrical objects and students' strength for understanding mathematics—, this article suggests that the theory-building process in mathematics education is a heteronomous and interdependent activity. From an enactivist perspective, the authors highlight that theories evolve through progressive individualization, encounters with theories in various domains, and subsequent collaboration. In particular, collaborating with theories from other fields can generate new questions and offer valuable conceptual perspectives. This collaboration is vital in developing interdisciplinary research in mathematics education.

Keywords: Progressive individualization; Collaborative and emergent progress; Case study

INTRODUCTION: IMPLICIT ASSUMPTIONS IN THEORY BUILDING

In contemporary educational research, it is crucial to address the issue of inclusion. The concept of inclusive mathematics education proposed by Kollosche et al. (2019) involves various political, social, cultural, and personal aspects, on the one hand, as mathematics has been discussed as a specific moment of exclusion processes for a long time. On the other hand, in the Japanese context, mathematics education research is often just regarded as an area of subject pedagogy that anybody can access equally, and its contributions are evaluated based on whether they provide practical suggestions for addressing educational problems (Kageyama, 2022). In Japan, one of the current crucial aspects to consider is diversity. Rather than distancing oneself from this issue because research interests differ, providing some empirical insights or explanatory frameworks can prevent overlooking or underestimating theories in mathematics education. The concept of diversity has now moved beyond the notion of physical disability. It is seen as a characteristic of the individual, leading to ethical critiques of general teaching methods in the classroom, outdated teaching materials, and the social environment (Lambert, 2015, 2019). If mathematics education research is not updated to address these contemporary challenges or if people do not specialize in advanced mathematics education, they often seek therapeutic, pragmatic coping solutions to ongoing problems. Moreover, this

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domain-specified approach, without any dialogue among specialized knowledge, runs the risk of not encourage reflections on authentic mathematical practices or the accumulation and transmission of empirical knowledge.

Numerous theories exist in mathematics education research. The maturity of a research field is evident in the growth and development of unique theories, accompanied by technical terms that enable researchers to identify, describe, and understand phenomena. Despite these benefits, the actual process of theory building is often omitted from academic publications, putting readers at risk of overlooking the original context and assumptions behind the theories. The axiological dimension of theorizing (Sinclair & Herbst, 2024) has discussed the values formed by researchers in theory building, considering that research findings do not necessarily reflect the intentions of the various participants involved, such as researchers and working practitioners. When seeing theory building of mathematics education as a socio-spatial practice performed at the collective level, we can draw spatial metaphors of theorizing such as pyramids, trees, mapping, and so on (Sinclair & Herbst, 2024), while ignoring the various attributes that the original problem initially includes highlighted by observing the historical background in which the theory was formed, what problems it addresses, and under what circumstances. In this regard, we suggest the following implicit assumptions about people.

Firstly, although not all theories are applicable, as mathematics education is a humanistic science built on data obtained from individuals, theories in this field are evidence-based, at least regarding human educational practice. Practice generally refers to the behavior, interactions, and collaborative work of individual students and teachers, as well as the collective behavior identified in the classroom. Some prior theories in mathematics education are based on the assumption of ideal students who are considered healthy, perceptually competent, obedient, positive about mathematics, skilled at communicating, and proficient in using media in the classroom. For example, according to Källberg & Roos (2025), a student perspective in mathematics education research, such as "feelings, experiences, and views", "evaluating interventions", and "validating effects", is often treated as relying on "regular" students. Marginalized students and those at different attainment levels are rare; therefore, diverse backgrounds and aspirations are sincerely acknowledged, despite the diversity of students present in educational settings. There is some research focusing on low-attaining students and reports of efforts to address them (Hodgen et al, 2024) and research on how these students feel about the special instructional methods proposed so far (Macchioni & Baccaglini-Frank, 2025), but low attainment in mathematics is only a limited phenomenon when viewed through the filter of existing assessment tests and criteria. Therefore, this research does not necessarily address what students of inherently diverse backgrounds think before they are assessed. Furthermore, considering a suggestion that research methods and the direction in which research findings are generalized differ depending on the disability status of the subject students (Lambert & Tan, 2020), the contributions of these theoretical findings will also be hypothetically limited to the ideal world in which these people exist. Thus, we must consider building an alternative mathematics education theory for people with special needs or specific social and cultural backgrounds.

Secondly, research from a metaphysical or cultural perspective assumes the existence of an unspecified majority, observing the phenomenon of mathematics education apart from the attributes of individual people. For example, in complex systems science (Mitchell, 2009), the characteristics of those of individual aggregates do not necessarily correspond to the individuals, nor to the findings of disability studies focusing on persons with disabilities (Davis, 2016). Furthermore, mathematics is not necessarily universal, and the existence of indigenous mathematical knowledge tailored to specific regions worldwide has begun to attract attention for future educational practices (Lipka et al., 2020; Chronaki & Lazaridou, 2023). In line with this trend, there have been recent challenges in critically examining Western-centered and Euclidean-based mathematics education. For example, Gerofsky et al. (2018) express concerns that straight lines and grids are becoming implicit metaphors that support social infrastructure under the guise of rationality, thus threatening our realm of thought where "straight is good". The foundation of *major mathematics*, as defined by de Freitas and Sinclair (2020), caters to a majority with blurred boundaries. Specifically, diverse minorities who rely on *minor mathematics*, which relates to indigenous mathematics and is often erased by state-sanctioned curricular images of mathematics, may face violence when entangled with the majority. Thus, when theories of mathematics education research are generalized for an unspecified majority, or when they focus on limited but seemingly universal regional or cultural characteristics beyond the unspecified majority, we need to be cautious when applying them.

Although the scope of contemporary mathematics education research has expanded, not all theories are based on the above assumptions–ideal people and an unspecified majority. In the growing and developing field of mathematics education research, particularly in the dynamics of theory building, such assumptions might not be initially apparent in the first place. In this article, we pose the research questions below to characterize an actual theory-building in relation to the assumptions.

RQ1: How does a theory of mathematics education come to include assumptions that are neither included nor implicit at its initial formulation?

RQ2: What are the characteristics of building a theory of mathematics education that comes to include these assumptions?

To address these research questions, this article draws on the local theory-building process in which the authors had been involved to demonstrate that theory building seems a heteronomous and interdependent activity. While adopting an auto-ethnographic method (Ellis & Bochner, 2000; Jones, 2005) to specify the historical background of the theory-building process by referring to some collected documents and data such as well-designed lesson plans, the authors' field notes of observed practices, and oral communications with educational participants, the authors will describe the dynamics of theory building as starting from solving a local, situated problem or providing an explanation of phenomena, and then encountering another related theory, leading to relativization or subsumption. A distinctive feature of this methodological approach is that it allows us to examine the actual building process, which is often omitted from published articles, as well as the intentions and emotions that accompany it, as a resource for our analysis.

If the suggestions through the analysis are reasonable and acceptable, should mathematics education research be deconstructed to make it worthwhile for all educators by reflecting on these assumptions, or can it progress flexibly to the next building stage? Accordingly, this article proposes a flexible shift and collaborative complementation among theories, rather than merely applying to other fields, as a contribution to mathematics education theories. As follows, the article proposes two directions for theory building—progressive individualization, and collaborative and emergent progress—by exploring the dynamics of theory building through two case studies: geometrical knowing and learning mathematics by students with disabilities.

CASE STUDIES: TWO DIRECTIONS OF THEORY BUILDING

Theory-building proceeding in the case of geometrical knowing

From the perspective of enactivism (Varela et al., 1991; Maturana & Varela, 1992), the first author proposed three criteria for student's knowing geometrical objects in the mathematics classroom–theoretical, possible, and actual–to seek the research question "what criteria do students adopt to generate an object and determine whether it is geometrical or not?" (Kageyama, 2018). This question emerged through ongoing discussions with the collaborative teacher to understand various students' geometrical meaning. Theoretically speaking, the geometrical world is co-defined between the different media in the classroom (e.g., artifacts and concrete objects with pedagogical intent), diagrams on the blackboard and in notebooks, as well as perceptions and actions by learners and teachers from different perspectives (de Freitas & Sinclair, 2014). The geometrical object is significant within the world, and it exists in connection with the participants' ways of knowing. According to the findings from mathematics classroom observations, the first way of knowing a geometrical object is theoretical knowing. It depends on the technical and symbolic language used in the mathematical world. The second way is actual knowing, preferring bodily movements and sensations in the physical world. The third is possible knowing, which is a medium between actual and theoretical knowing. It enables us to perceive and understand events and objects that cannot be directly referenced through bodily movements or biological perception, using metaphors of movement and language as tools.

The core idea of adopting enactivism in geometry education is that it incorporates what brings forth the significant worlds that realize geometrical discourse (Varela et al., 1991; Maturana & Varela, 1992) through recurrent interactions in which differences in students' bodily actions, sensations, and intentionalities are problematized, thereby advancing their thematic discussion. In doing so, it generates system-based arguments for group activities, as presented by Kieren and Simmt (2009). The significant world for participants of the classroom establishes operational norms that guarantee the validity and significance of geometrical objects produced through various types of actions. Supported by enactivist interpretations of data from several hours of elementary school mathematics lesson observations, the insight was gained that, in a physical environment such as the classroom or surrounding materials, the objects dealt with by the teacher and students were geometrical when they possessed these three aspects simultaneously (Kageyama, 2018). Speaking theoretically, the last concept is based on isomorphism. Specifically, the change and movement of objects explained using gestures and natural language in the actual world, and the transformation of objects described using mathematical expressions and symbols in the mathematical world imply mathematical isomorphism with reflexive, symmetric, and transitive relations. This is because by focusing only on the relationship of the corresponding points between the start and goal figures before and after the gradual change, the time of change can be ignored. The possible worlds are those that mediate actions in both other worlds, actual and theoretical. Because actions are finite for those whose bodies and time are finite, it is not possible to manipulate objects anywhere in the actual world. However, by assuming the infinity and homogeneity of plane and space, we can explore the movement of objects using visual and operative patterns. Conversely, visual and operative images of objects that exist in the mathematical world through mathematical expressions make sense in the possible world, not in the actual world.

In relation to RQ1, our local theory–the three criteria for knowing geometrical objects in the classroom–

constitutes the initial stage of theory building, prior to encountering or including other assumptions. Then, this moves to relate it to the findings of mathematics education research, clarifying its originality and developing its pedagogical significance, an early process in the theory-building process. For example, theoretically speaking from the viewpoint of *structural coupling*–referring to the recurrent interactions through which two systems become congruent (Maturana & Varela, 1992; Reid & Mgombelo, 2015), as emphasized by enactivism–concrete objects and bodily actions are inextricably intertwined. This leads to the emergence of qualitative questions and emotions that arise from the interaction between the environment and the body in a classroom environment. Moreover, problematizing differences, leading to the generation of different knowing criteria, shares similarities with the level theory of geometrical thought (van Hiele, 1986). This theory has developed by examining differences in students' language use, as such differences reflect corresponding differences in levels of thinking (Battista, 2007). However, Kageyama (2018) utilizes scientific and philosophical theories, such as cognitive science and embodiment theory, along with specialized techniques (e.g., high-resolution video cameras and instantaneous text data transcription) that have evolved concurrently over time. Kageyama (2018) thus argues that informal mathematics constructed by children is not strictly limited to word use and provides insight into the inherent diversity of mathematics, demonstrating that various forms of mathematics can coexist. Cognitive science and enactivism are heterogeneous in mathematics education in the sense that they have different research interests and directions and have not always been cross-referenced. Nonetheless, despite their potential usefulness, they could play important roles in theory building to explain and emphasize the multiplicity of features of geometrical practice implemented by students and teachers.

In the next phase of our local theory building, limited to the observed classroom activities and related literature reviewed, we gradually approached the so-called grand theories in mathematics education. One of these is the "three worlds of mathematics" proposed by Tall (2013). This is significant because it integrates traditionally separate discussions on mathematical cognitive development into a long-term developmental theory from the viewpoint of mathematics education. And the growth of mathematical thinking implies that the difficulty in understanding mathematics lies not only in a lack of individual ability, but also in an awareness of the changing nature of "thinkable concepts" (p.51) that can be dealt due to knowledge compression and of the different ways of studying, such as thought experiment and symbolic manipulation, to integrate the three worlds of mathematics (pp.402-407).

The three worlds proposed in this theory (embodied, symbolic, and axiomatic) are similar to the three criteria for knowing geometrical objects in terms of labels in our local theory (Kageyama, 2018). Still, they do not correspond conceptually. For example, "embodied" in Tall's theory is used in a different sense from concrete behavior in bodily cognition. Moreover, "possible" in our theory does not refer to any of the worlds in Tall's theory, since it refers to the state of being aware by inference of events that cannot be directly grasped perceptually. The "axiomatic" in Tall's theory is a world consisting of theorems and axioms for their statements that are deductively proved to build a mathematical knowledge structure. This stage does not fully overlap with our "theoretical," which is a range of school mathematics. Notably, however, Tall's theory considers the temporal transformation of the semantic understanding of mathematical objects, and that similar progress is observed in a short period (even though both progress and regression occur), with abstract concepts such as isomorphism being formed through various cognitive forms. Thus, once again, the ideas of this grand theory can be incorporated into the structuring of complex events that occur in mathematics classrooms, the analysis of mathematical concepts in terms of learners' cognitive forms, and the design of long-term mathematics instruction. Such a journey involves comparing and contrasting theories originating from different backgrounds concerning problem posing and its solving, or an intention to present an alternative explanation framework of observed phenomena (Bikner-Absbahs & Prediger, 2010; Prediger et al., 2008); however, the goal does not necessarily lead to the complete integration of one theory into another. Instead, it leads to a relative characterization of theories through contrastive reflection that considers the similar or different backgrounds of the theories. It involves looking for key conceptual counterparts in each theory, (re)defining the aims and direction of theory building, and establishing relationships between theories. The progressive theory-building described above (i.e., building while maintaining tensions between a heterogeneous research area and a grand theory) can be seen as *a progressive individualization following contrasting,* by referring to a networking strategy (Bikner-Absbahs & Prediger, 2010; Prediger et al., 2008). This is because it involves an iterative conceptualization of key ideas (e.g., preparing appropriate terminologies and their networks), complementation, and incorporation of related theories (i.e., interpreting prior theories from the perspective of developing the theory to identify the similarities or differences between them, vice versa). According to the metaphorical characterization of the dynamics of theory development by Scheiner & Bosch (2025), this proceeding is a phenomenon that is recognized through the dynamic movement of both centripetal and centrifugal forces in the inter-theoretical dialogue. Just as proceeding is not linear due to both forces, it can be cyclic, with some periods of research activity engaging in differentiation and others oriented towards integration. The direction of research activity will be dictated by the coherence of current research interests with those initially faced, such as which phenomena to increase explanatory power for, what normativity to bring to the theoretical discourse.

SCOPE OF A THEORY IN THE CASE OF MATHEMATICAL COGNITIVE DIVERSITY

Zoom in and out of the image as the observer's behavior for mathematics learning

In mathematics education research as a human science, some issues arise regarding how much to consider not only diverse people as research objects, but also the particularities of each individual. The relationship between diversity and particularity is related to the resolution of the intentional object, which becomes clearer when seeing people either as individual autonomous actors (*zooming in* on the scope of the theory to reduce the angle of view to describe their movements in detail) or as groups that form a culture (*zooming out* on the scope of the theory to increase the angle of view to provide a comprehensive description of their movements).

Reducing a particular person's condition to physiological data and then interpreting the data by comparison with an assumed person's typical condition might be a general *zoomed-in* method. For example, Lake and Nardi (2023) used physiological data to represent changes in teachers' emotions during lessons. They contributed to the development of guidance that promoted self-reflection by combining it with interview data. However, according to a recent systematic review of emotion research (Schoenherr et al., 2025), a body of research argues that emotions are learner-specific yet embedded in social contexts, thereby forming relationships with mathematical achievement. To understand how an individual's emotions arise through interactions with others within a group, it is necessary to *zoom out* from the individual. By gradually *zooming out*, the person's social rule-bounded state of multiplicity might be seen. Addressing this issue is a challenge to the assumption of "assuming an unspecified majority" mentioned at the beginning of this article, which corresponds to the different problem settings that can be grasped by *zooming in* and *out* on the subject.

Recognizing the diversity in mathematical cognition, which has recently been referred to as neurodiversity (Leikin, 2018), different approaches will not only lead to research efforts aimed at generalizing and expanding research findings, but also foster encounters with work that was previously considered to belong to other domains. According to Radford (2008), a research domain consists of a research question and a theory that informs it. Moreover, it refers to a specific area with principles and methods related to the question and theory. However, it is also important to consider the responsibility that recent researchers should have for modern scientific research conducted on behalf of citizens and the social impact of their research. It seems that the reducibility of research results is also a social criterion for research to be recognized and trusted by society. In this article, in addition to the view of mathematics education research as a dynamic practice (Prediger et al., 2008; Bikner-Absbahs & Prediger, 2010), the first author also considered the enactivist perspective, where knowledge emerges in an autonomous system (Maturana & Varela, 1992). Such a system emphasizes the co-defining relation of perception and action, characterized by recurrent dynamics, in which knowledge constructed through one activity provides the basis for the next. From this perspective, theory in mathematics education can be understood as a body of knowledge with potential generality (Schoenfeld, 2007)—that is, generality within which findings generated through this continuous movement can be validly applied.

For example, addressing challenges related to access and interaction in Universal Design for Learning (UDL; Hall et al., 2012) involves an engineering paradigm for the inclusive participation of students with disabilities in mathematical practices. While UDL considers support for students with disabilities as the starting point of its research journey, it is accompanied by an ideological shift in its background. It includes the endorsement of scientific findings such as brain science, support for teachers in practice, a guarantee of human rights, a social model-based view of disability, and a challenge to a narrow acquisitionist view of learning. However, UDL lacks ideas on how to adapt learning goals for students with severe difficulties, because it is designed to develop instructional tools and strategies that fit within a specific range of competencies, such as emphasizing relations among concepts (Krähenmann et al., 2019). Against this background, in UDL, mathematics is regarded as a tool set for an individual's life in society or as a set of knowledge to be formally learned in school, and the primary mission is to facilitate its acquisition effectively. In this sense, mathematics education research and UDL research are heterogeneous research areas, as they may have different meanings and expectations for the same words, such as mathematics and diagrams. However, they are related to the formulation of potential problems in mathematics learning practices and their improvement, which are revealed through the mediation of mathematics, but also sometimes by *zooming out*. Moreover, in UDL, when it comes to supporting cognitive networks, customizing how information is presented with appropriate media and explaining concepts through various presentation formats involves issues related to fixed interfaces. For mathematics education research to create a social impact, it might be crucial to examine whether mathematical cognitive development, which has traditionally been discussed separately, applies to students with disabilities, how well-designed artifacts can support mathematical practice, and what kinds of process transformations are triggered by introducing interfaces.

Students' strengths for understanding mathematics

Regarding the interface issues mentioned in the previous section, the second author of this article, who used to be a mathematics and special needs educator in an elementary school, has collaborated with special needs researchers, educational practitioners, and educational administrators on inclusive design in mathematics classes to summarize some useful findings (Matsushima & Era, 2021), drawing on previous mathematics education research documents such as Woodward (1982) and Hasegawa (2007). They reported that the following misconceptions were often found in measuring figures, regardless of whether disabilities were present or not.

- As the perimeter of the figure increases, the area also increases.
- There are formulas to find the area of each figure that are unrelated to each other; therefore, it is not possible to use the formula to find the area of one figure by using the formulas for other figures.
- When there is no height inside the triangle or when the base of the triangle is not horizontal, it is not possible to determine the height of the triangle.

Misconception is a conceptual tool in research programs based on constructivism (Sfard & Cobb, 2014), in which learners subjectively construct their knowledge. Adaptation to the environment through interaction is also considered an essential factor in knowledge construction. Therefore, in measuring the area of the given figures, the dimensions of the measuring units need to be identical for a quantity to be measured. To dissolve misconceptions, teaching methods should include the use of counterexamples and specific materials, such as geoboards, to enable learners to work independently. It is essential to make sense of mathematical objects by using an individual original way, regardless of the students' disabilities, as students' epistemological behavior is characterized by the adopted epistemological standpoint (Ernest, 2011). Therefore, rather than treating students with disabilities as having some ability deficit, it is necessary for us to go beyond focusing on operations in a general sense and guide them to construct their own mathematical meaning in a way that utilizes their strengths.

The following example of meaning formation in mathematics, which utilizes the strengths of such students, has been published in a practical research journal on special needs education (Ikeda, 2023). Student A was a fourth grader enrolled in a regular class and had an obstruction in memorizing a multiplication table. In Japan, the national curriculum deals with a multiplication table in the second grade of elementary school. In general, the goal of Japanese mathematics education is to enable students to memorize 81 types of multiplication facts, such as 1×1, 3×5, and 9×9, by humming them rhythmically. However, since Student A was unable to recite these number facts until at least the fourth grade of elementary school, he was diagnosed with considerable difficulty in learning mathematics. Based on the results of the Wechsler Intelligence Scale for Children (WISC-IV), Student A was characterized as having weak working memory, preferring simultaneous processing to sequential processing, poor writing skills, and a love of sports. Therefore, in this practical research, based on the actual situation of Student A, the practical supporter introduced an activity in which a Skip Counting Mat (Math & Movement, 2025), 50 cm wide and 6 m long, with answers to the multiplication table written on it, was laid out in the classroom. The student jumped on the mat five times per hour while reciting the multiplication table. When reciting the multiplication table, a multiplication expression card was placed next to the mat, making it easier for Student A. This activity was devised based on the following principles:

(1) To utilize Student A's strength in simultaneous processing, a mat that allowed the entirety to be seen visually was used.
(2) To utilize Student A's love of exercise, Student A was motivated to participate in the activity. To compensate for his weakness in working memory, multiple senses, including vision, hearing, and kinesthetic sense, were used to reinforce memory retention.
(3) Considering Student A's writing weakness, writing activities were not introduced.

As a result of introducing this activity and integrating bodily actions and numbers, Student A was able to memorize the multiplication table completely. After the lesson, the students commented, "I had never been able to remember my multiplication tables before, but I was able to memorize them by jumping for the first time. I am so happy" and "I was able to memorize them by jumping continuously". In a classroom for learning multiplication, it is probably rare for students to jump on a number sheet mat instead of writing mathematical expressions. However, teaching methods such as reciting numbers rhythmically or like a chant to memorize multiplication tables or experiencing the proportional relationship between the multiplier and product by stacking concrete blocks are popular in Japan, as described in many textbooks. Both Student A's jump and the "regular" students' recital are methods of embodying abstract mathematical objects through multiple bodily functions; thus, appropriate proposals involving mathematical meaning formation could be within the scope of mathematics education theory.

The case was presented in another research field, specifically special needs education, demonstrating that students who were uncomfortable with standard mathematics teaching methods could achieve the given learning goals by utilizing their strengths rather than being treated as lacking abilities. We could draw some implications concerning instructional design. The first is the setting of learning activities that utilize students' strengths. In this case, the setting of activities using the skip-counting mat leverages students' strengths in simultaneous processing and their love of exercise. Practical research on mathematics education that utilizes the results of cognitive tests such as Wechsler Intelligence Scale for Children IV (WISC-IV) is common in other research fields concerning special needs education but is rarely seen in mathematics education research. The second was the setting of learning activities using multimodal methods, including visual, auditory, and kinesthetic. Although mathematics education research has gradually emphasized design-based, multimodal learning methods (e.g., Abrahamson et al., 2023), there are still few.

This case concerns numbers and calculations, but also relates to the knowing criteria: theoretical, possible, and actual (Kageyama, 2018). Utilizing the students' strengths will encourage them to engage in learning activities and create possible ways to generate mathematical concepts. In addition, setting up multimodal activities involves establishing actual cognitive methods according to the individual. According to the three criteria for knowing geometrical objects described above, forming various but coherent meanings for multiple worlds, such as the physical and practical phenomena that occur through concrete manipulations; exploring possibilities to consider all cases; and insights into mathematical structures and descriptions, is critical. To enable different student behaviors within significant worlds of quality, the learning environment design and teacher interventions must fit the criteria of each world.

While the above discussion is based on zooming in on individual behavior, we can zoom out to consider areas beyond our direct perception. When the scope of mathematics education research is magnified to a global scale, which humans cannot perceive directly, we face even greater challenges beyond individual cognitive characteristics. Climate change and population issues are typical examples (Barwell, 2013; Mikulan & Sinclair, 2017). Such cases require helpful suggestions for both traditional and growing theories in mathematics education. Still, in most cases, they are seen as future needs for mathematics education research because they cannot be covered by the pairing of the event in question with the submitted theory (Coles et al., 2024). As this research needs to relate to new research questions concerning the socialization of research findings, we suggest that these cases are ongoing challenges. However, *collaborative and emergent progress can be achieved through the encounter of theories from different fields* as another direction for theory-building. In reflecting on research areas that have evolved in isolation, theorems established based on specific individual or collective tendencies should be generalized to include other particulars, which will reveal the relative characteristics of each other's research challenges. In addition, during the development of an idea into a theory in mathematics education research, there is a point at which its inclusiveness and relationship to concurrently developing theories become problematic. Addressing this issue presents an opportunity to reflect on exploratory theory, identify critical elements, and close the gaps in cohesive theory. In the following section, we examine two proposed theory-building approaches that emphasize the process and scope of a theory.

TWO DIRECTIONS FOR THEORY BUILDING

In the previous sections, we proposed two directions for theory building:

- Progressive individualization following contrasting
- Collaborative and emergent progress from the encounter of theories from different fields

Considering the first building direction, in a semiosphere comprising multiple studies (Radford, 2008), it is important to point out that the questions and claims may appear similar, but they have different interests and starting points. Considering the characteristics of mathematics education research as the cyclic growth and self-generation of tasks from an enactivist perspective, and the social contribution that research itself should have, each theory should inherently be autonomous. The further the theoretical movement progresses, the more differentiation of theories will occur, and the more complex the semiosphere will become. As a result, we will have many theories according to the number of starting points of research; however, it is beneficial for the growth of a theory to shift its interest to another related theory or finding. Moreover, it should evaluate the original interest or question from its perspective, resulting in the integration, specialization, and generalization of various theories. The progressive individuation following contrasting in this article represents a flexible shift just distinct from integration, leading to semi-grand or handmade theory building. This proceeding does not lead to an unlimited increase in theory. Rather, it suggests that accumulating such shifts can lead to a comprehensive maturation of mathematics education research theory.

Even if the original research interests and subjects overlap, multiple theories may emerge as if they had been created. It includes the desire to clarify the typical, or complex developmental process of people's mathematical understanding (e.g., Tall, 2013; Pirie & Kieren, 1994; Pirie & Martin, 2000); the desire to assess special mathematical strengths by departing from the narrowly defined deficit model to design an interventional instruction (e.g., Hunt & Tzur, 2022); the desire to design effective mathematical instruction based on a alternative paradigm (e.g., Abrahamson et al., 2020; Abrahamson et al., 2023); and different research backgrounds and routes of theory development depending on who is involved in the research activity based on the second building direction. In the last case, when the expected contributions to the theories are the same, those involved in the theory will attempt to address them as complementary or strengthening one or the other. For example, in the encounter between a special type of mathematical cognitive theory (e.g., Clark et al., 2023) and UDL (Hall et al., 2012), whose scopes are different with each other, some key terms that remain incompletely conceptualized are used, such as *obstacles* and *loads*, in addition to specialized research questions. Making mathematical media accessible as support does not necessarily reduce the load of facilitating mathematical thinking or translating representations within diverse representational systems. The need to consider the interactive relationship between special cognitive properties, mathematical essences, and their embodiment, mediation, and media draws from collaborative, dialogic work between mathematics education and special needs education research. The study of meaningful communication, representations and symbols in traditional mathematics education research (e.g., Pimm, 1995) has a long history in which the relationship between mathematical concepts, meaning, and symbols has been emphasized. We must consider the special situation where claims are made "from outside mathematics education research" that certain mathematical representations may be an obstacle rather than a load for learning for users and interpreters of representations, and move on to more inclusive, effective theory building.

Focusing on the issue of inclusion, it is notable that key terms are often used within each other's discourse; however, they do not always have the same meaning and may vary according to their significance as stated by referring to UDL. Therefore, in everyday practice, key issues are not always identified autonomously. Theoretical statements have been generated using diverse questions that transcended various domains. It may sound paradoxical, but the deliberate non-systematic generation of questions about what key terms can be, what they mean, and how important they are will lead to a set of theoretical statements being continuously generated. Assuming that those not involved in theory building or use are included, these statements must be identified in a process that maintains certain tension. If mathematics education research is seen as interdisciplinary, identifying conceptually important aspects while maintaining tension will contribute to the continuous growth of the theory.

CONCLUSION

Consistent with the view that mathematics education practice is a dynamic activity that involves a dialogue between theories whether or not the direction is toward proposing an integrated theory (Bikner-Absbahs & Prediger, 2010; Scheiner, & Bosch, 2025), and characterizing real research activity from the standpoint of enactivism as a continuous one self-generating any questions, the authors demonstrated that a theory itself is inherently drawn through a cyclical movement involving question generation and response. Through this movement, mathematics education theory will be subject to criticism from within and outside its boundaries, becoming a homegrown theory. The movement does not expect to make transdisciplinarity in mathematics education (Jao & Radakovic, 2018), nor does it attempt to construct a common conceptual framework that can span various disciplines; instead, it suggests that each theory, with different problematic situations and research interests, is inherently both autonomous and heteronomous in nature. Here, in detail, participants in the mathematics education research cycle with diverse interests construct a network of heterogeneous and interdependent relations and engage in cyclical theory-building activities. However, the complexity of collective activities may allow an abundance of statements that cannot be considered theoretical. At this stage, progressive individualization and collaboration can guide the development of a coherent theory.

This article proposed two directions of theory building through case studies in which the authors were involved. The proposition includes several hypothetical implications for claim coherence: the autonomous and heterogeneous growth of theories that occurs through contrast and relativization, the relationship between increasing complexity and integration or coherence orientation, and the different significance given to the same key terms across different theories. Challenging these issues is beyond the scope of this article, which aims to provide possible paths of theoretical proceeding, but they must challenge for future research. For example, what are the students' strengths, and should they be conceptualized in relation to effective mathematics learning? What is the difference between strengths and preferences? Should strengths-based instructional intervention be designed to dissolve well-known mathematics dogmas (Coles & Sinclair, 2022) such as abstraction and rigor? For inclusion-oriented transformative mathematics education, theory in mathematics education must continue to grow sustainably with diverse participants.

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